

Electric Circuits

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1 What is current?

Electric current is the measurement of how much charge passes a certain point during some time. Current is not the same as the speed at which some amount of charge is moving because speed varies greatly depending on which particle is examined. The speed and direction of each charged particle varies, but the number of charged particles passing through a certain point varies only negligibly. Current (I) is the number of Coulombs per time and is measured in Amperes (Amps).

$$I = \frac{dq}{dt}$$

1.1 What makes the charge move?

Just as a pressure difference makes water move through a pipe, a voltage difference (also known as an electric potential difference, an electromotive force, an emf, or ε) makes charge move from one point to another. Positive charge will move from the high-voltage side to the low-voltage side. A battery or generator can provide this voltage difference.

1.2 What is the relationship between current and speed?

The circuit containing positive charges (electron holes) that move faster has a greater current because it can move more charge in a certain amount of time. The net movement of the charged particles is the **drift velocity**.

Because current depends on the number of charged particles per volume, the quantity of charge, the cross sectional area of the wire, and the velocity of the charges, we obtain another equation for electric current.

$$I = nqv_dA$$

Where n is the density of the charge (the number of charged particles per cubic meter of volume), q is the quantity of the charge, v_d is the drift velocity of the charged particles, and A is the cross-sectional area of the wire.

2 Electric Current Density

Electric current directly depends on cross sectional area. Decreasing the cross sectional area decreases the current measured. However, the density of the current remains the same. Current density is the ratio of current to cross-sectional area. The units are Am^{-2}

$$\vec{J} = \frac{I}{A} = nq\vec{v}_d$$

Current is the scalar product of current density and the area vector because current is maximized when the area vector is parallel to the current density vector.

$$I = \vec{J} \cdot \vec{A} = JA \cos(\theta) = \int \vec{J} \cdot d\vec{A}$$

2.1 Resistivity (ρ)

The electric field created by a voltage difference is used to move the charged particles through the circuit. The electric field and the electric current density differ from each other by a factor known as the resistivity, which varies by material. The greater the resistivity, the lower the electric current density.

$$\vec{E} = \rho \vec{J}$$

3 The Direction of Electric Current

Current does have direction, but current is **not a vector quantity**. Vector quantities obey the laws of vector addition and can have vector components; however, electric current direction is relative to the charge carriers, predefined by the circuit. Current does not obey the laws of vector addition.

However, Ben Franklin's convention bases the current on the direction of positive charges, even though the negatively charged electrons are actually moving. You can think of current as electron holes moving through the circuit. The direction of conventional current is the direction in which the positive charge carriers are moving.

4 Non-Constant Current Density

Suppose that a wire with radius R has an electric current density that varies with radial distance r as $J = ar^2$, where a is a constant. What is the current through the outer portion of the wire between $\frac{R}{2}$ and R ?

This problem is fairly trivial. Recall the formula for current, note that the angle between the current density and the area vector is 0, and integrate.

$$I = \int J \cdot dA = \int J dA = \int_{R/2}^R Ar^2 2\pi r dr$$

5 Simple Circuits

A simple electric circuit is composed of electrical loops which can include batteries, resistors, light bulbs, capacitors, inductors, switches, ammeters, and voltmeters.

An open circuit is any circuit in which charges are not able to flow. It has an open switch: it isn't complete.

6 Resistance

Resistance, R , measured in Ohms (Ω) is the opposition to the movement of charges through a circuit.

Ohm's Law states that $V = IR$. If there is no resistance in the circuit, and there is a voltage difference, then current will diverge to infinity.

A resistor is like a bottleneck. It clogs the flow of charges, forcing charges to cram together and slowing the overall current, but increasing the speed of the charges within the resistor as it forces them to cram together. The cramming together of charged particles dissipates kinetic energy as heat and light.

Because resistors convert the potential energy of the charges into kinetic energy, there is a decrease in voltage (a voltage drop) across the resistor. Current is constant across a resistor, but voltage decreases.

6.1 Resistance and Resistivity

Resistivity is a fundamental property of materials that characterizes its ability to impede the flow of current. Resistivity determines how well a material can resist the flow of current.

The inverse of resistivity is conductivity.

Resistivity is the ratio of Electric Field to Charge Density. Recall that

$$\vec{E} = \rho \vec{J}$$

The electric field is the voltage difference per distance, and the current density is the current per area.

$$\frac{\Delta V}{L} = \rho \frac{I}{A}$$

Current is equal to voltage per resistance ($V = IR$).

$$\frac{\Delta V}{L} = \rho \frac{V}{RA}$$

Finally,

$$R = \frac{\rho L}{A}$$

Temperature's affect on resistivity can be neglected unless told otherwise.

6.2 Equivalent Resistance

Equivalent Resistance is the total resistance for a section of a circuit.

When resistors are combined in series, total resistance increases. When resistors are combined in parallel, total resistance decreases.

$$R_{equiv,series} = \sum R_i$$
$$\frac{1}{R_{equiv,parallel}} = \sum \frac{1}{R_i}$$

6.3 Solving Circuits with Ohm's Law

6.3.1 Limitations of Ohm's Law

Ohm's law does not apply when

1. the resistance of a material varies with voltage or current (a Non-Ohmic Material)
2. temperature does not change (because resistance varies with temperature)
3. using an AC circuit because voltage and current vary
4. at very high frequencies

6.3.2 Solving Circuits

First, combine all resistors to find the equivalent resistance. Compute the initial current as $I = V/R$. Then, for each individual resistor, recall that in series, current is the same, and in parallel, voltage is the same. Change either Voltage or Current based on the Resistance of the resistor.

7 Electric Power

Power, P , measured in Watts (W) is the rate of energy dissipation or transformation.

Work is the transformation of energy from potential to kinetic.

Power is the derivative of Work with respect to time.

$$P = \frac{W}{t} = \frac{qV}{t}$$

$$P = IV = \frac{V^2}{R} = I^2 R$$

When **current is constant**, such as when considering resistors in series, use

$$P = I^2 R$$

When **voltage is constant**, such as when considering resistors in parallel, use

$$P = \frac{V^2}{R}$$

7.1 Example Problem

When connected to a battery in series, which will glow brighter: a 40-Watt bulb or a 100-Watt bulb?

The 40-Watt bulb will glow brighter because if current is constant ($P = I^2 R$), then Resistance is inversely proportional to power. Therefore, a 40-Watt bulb has a higher resistance, and therefore, it disperses more light energy and glows brighter.

When connected to a battery in parallel, a 100-Watt bulb will glow brighter because it has a lower resistance and allows more current to pass through it.

8 Non-Ideal Batteries and Circuits

In an ideal battery, the Terminal Voltage ΔV is the same as the EMF ε .

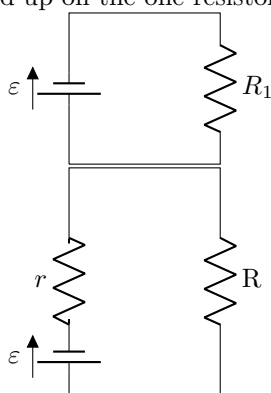
In a non-ideal battery, a battery has some internal resistance, so the Terminal Voltage is smaller than the EMF.

$$\Delta V = \varepsilon - Ir$$

8.1 Experimentally Determining if a Battery or Wire is Ideal

Construct a circuit with a resistor and a battery. Use a Voltmeter to measure the voltage across our load resistor, then vary the resistance of the resistor and repeat.

If our battery were ideal, the resistor should have the same voltage across it regardless of the resistor's resistance because all of the voltage difference created by the battery would have to be used up on the one resistor in the circuit.



To determine the internal resistance of the battery, use $V = IR$ to compute the total current I .

$$\varepsilon = I(r + R)$$

$$I = \frac{\varepsilon}{r + R}$$

Then, compute the voltage across the resistor.

$$V = IR = \frac{\varepsilon R}{r + R}$$

$$\frac{1}{V} = \frac{r + R}{\varepsilon R} = \frac{r}{\varepsilon R} + \frac{R}{\varepsilon R}$$

$$\frac{1}{V} = \frac{r}{\varepsilon R} + \frac{1}{\varepsilon}$$

Then, plot $\frac{1}{V}$ as a function of $\frac{1}{R}$ to obtain $\frac{r}{\varepsilon}$ as the slope of the line and $\frac{1}{\varepsilon}$ as the y -intercept.

9 Ammeters and Voltmeters

Ammeters measure current at a specific point in a circuit. Ammeters must be connected in series. Ideal ammeters have no resistance.

Voltmeters measure voltage and must be connected in parallel. Ideal voltmeters have infinite resistance so that no charge flows through them.

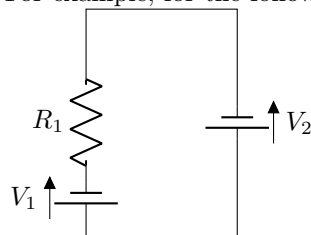
10 Kirchhoff's Loop Rule

Recall $\Delta U = q\Delta V$

If a single charge makes a complete loop around the circuit, conservation of energy says $\Delta U = 0$. Therefore, around an entire circuit, $\sum \Delta V = 0$.

Begin by choosing a starting point and sketching the current from that point. Then, across each battery, add the voltage, and across each resistor, subtract the voltage.

For example, for the following circuit, start in the bottom left corner.



Then,

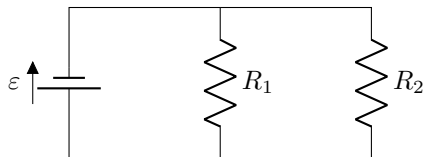
$$V_1 - R_1 - V_2 = 0$$

11 Kirchhoff's Junction Rule

Any charge that goes into a junction must come out of that junction.

$$\sum I_{in} = \sum I_{out}$$

For example, in the diagram below, the current through R_1 plus the current through R_2 must equal the current through the battery.



12 Solving Complex Circuits

Step 1: Determine a starting point

Step 2: Sketch the current

Step 3: Sketch the loop

Step 4: Apply Kirchhoff's Rules

Traverse each loop, adding voltage from batteries and subtracting voltage across resistors. Each battery will add a boost to the voltage, and each resistor will take away a little bit of that voltage.

13 Resistor-Capacitor (RC) Circuits

A capacitor is a set of two plates, the function of which is to hold charge. The **capacitance** (C), measured in Farads (F), is the amount of charge it can hold. If the size of the plates increases, then the capacitance increases because there's more room for charge. If the distance between the plates increases, then the capacitance decreases because the electric field decreases with increasing distance.

$$C = \frac{\epsilon_0 A}{d}$$

13.1 Capacitors in Series and in Parallel

When in **series**, capacitors will accrue the same voltage difference across it. Between capacitors, however, charge must go from positive to negative, creating neutral regions. Effectively, by placing the capacitors in series, we've spaced the plates farther apart because the spacing between the plates of the two capacitors adds together. As a result, adding more capacitors will decrease the amount of charge that can be stored.

Conversely, for capacitors in **parallel**, we increase the effective surface area of the capacitor without increasing the distance between the plates. By increasing the number of capacitors, the number of places to accumulate charge increases, thus increasing total capacitance.

$$\frac{1}{C_{equiv,series}} = \sum \frac{1}{C_i}$$
$$C_{equiv,parallel} = \sum C_i$$

To compute total charge, compute total voltage difference times total capacitance.

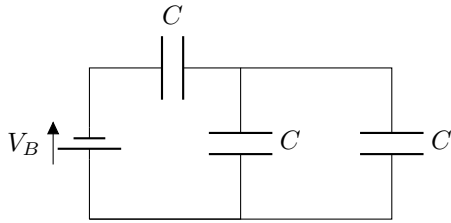
$$Q = VC$$

In series, capacitors have the same **charge** as the entire circuit because the current stays constant.

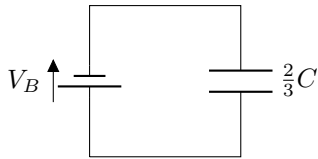
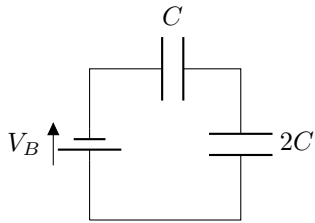
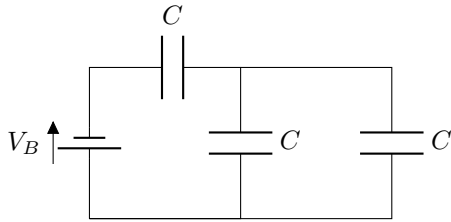
In parallel, capacitors have the same **voltage** as the entire circuit.

13.2 Sample Problem

Consider the following circuit.



Combine the capacitors to find the total equivalent capacitor.

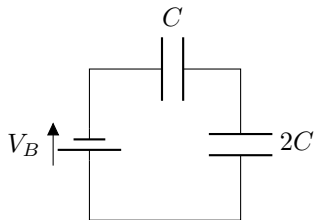


$$C_{eq} = \frac{2}{3}C$$

Using $Q = VC$, compute the total charge on that capacitor.

$$Q_{tot} = V_{tot}C_{eq} = V_B \frac{2}{3}C$$

Because the charge of a capacitor is the same in series, we work our way backwards.



$$Q_C = Q_{2C} = \frac{2V_B C}{3}$$

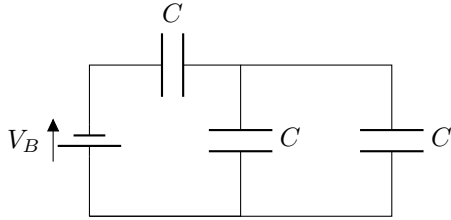
Using $Q = VC$ for each of the capacitors C and $2C$ in the second diagram, we know that the charge on both of these two capacitors must be the same as the charge on the equivalent capacitor. We can find the voltage across both the C and the $2C$ capacitors.

$$V = \frac{Q}{C}$$

$$V_C = \frac{2V_B C}{3} \frac{1}{C} = \frac{2V_B}{3}$$

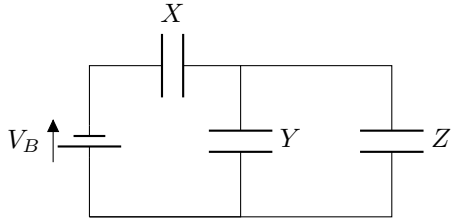
$$V_{2C} = \frac{2V_B C}{3} \frac{1}{2C} = \frac{V_B}{3}$$

Finally, because we've determined the voltage across the $2C$ capacitor, and because voltage is the same in parallel, we can find the charge across each of the parallel capacitors.



$$Q = \frac{CV_B}{3}$$

After solving the problem, we have



$$C_X = C_Y = C_Z = C$$

$$Q_X = \frac{2CV_B}{3}; V_X = \frac{2V_B}{3}$$

$$Q_Y = Q_Z = \frac{CV_B}{3}; V_X = V_Y = V_Z = \frac{V_B}{3}$$

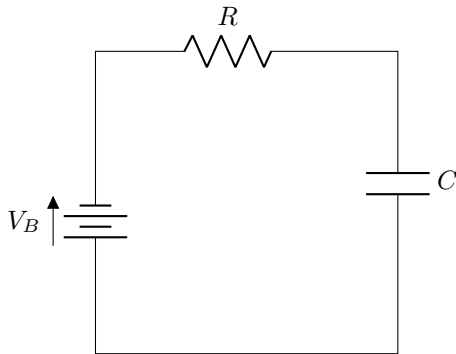
Sorry that I can't put these labels on the actual diagram.

13.3 Summary of Capacitor Laws

Capacitors in Series	Capacitors in Parallel
Current has one path it can travel	Current has multiple paths it can travel
Current and Charge are the same	Voltage difference is the same
Capacitance decreases (add reciprocals)	Capacitance increases (add)

13.4 Kirchhoff's Loop Rule for RC Circuits

The voltage across a resistor is $V = IR$. The voltage across a capacitor is $V = \frac{Q}{C}$.



Using Kirchhoff's Loop Rule, we find that

$$V_{battery} - V_{resistor} - V_{capacitor} = 0$$

$$V_B - IR - V_C = 0$$

Remember that $Q = CV$ and that $I = \frac{dQ}{dt} = C \frac{dV_C}{dt}$

$$V_B - RC \frac{dV_C}{dt} - V_C = 0$$

$$-RC dV_C = (V_C - V_B) dt$$

$$\frac{dV_C}{V_C - V_B} = -\frac{dt}{RC}$$

$$\int_{V_{C_0}}^{V_{C_t}} \frac{dV_C}{V_C - V_B} = -\int_0^t \frac{dt}{RC}$$

$$[\log(V_C - V_B)]_{V_C=0}^{V_C=V^t} = \left[-\frac{t}{RC} \right]_{t=0}^{t=t}$$

When charging the capacitor, we evaluate the left-hand side integral from zero initial charge on the capacitor to some charge at time t . If we were discharging the capacitor, we would integrate from the voltage of the full capacitor and discharge it.

$$\log(V_t - V_B) - \log(0 - V_B) = -\frac{t}{RC}$$

$$\log\left(\frac{V_t - V_B}{-V_B}\right) = -\frac{t}{RC}$$

$$\log\left(\frac{V_t}{-V_B} + 1\right) = -\frac{t}{RC}$$

$$\frac{V_t}{-V_B} + 1 = \exp\left(-\frac{t}{RC}\right)$$

$$\frac{V_t}{V_B} = 1 - \exp\left(-\frac{t}{RC}\right)$$

We have found the expression for the voltage across a charging capacitor.

$$V_t = V_B \left(1 - \exp\left(-\frac{t}{RC}\right)\right)$$

Using $Q = CV$, the charge across the capacitor as a function of time is:

$$Q_t = CV_B \left(1 - \exp\left(-\frac{t}{RC}\right)\right)$$

To obtain the current and voltage across the resistor as a function of time, recall that $I = \frac{dQ}{dt}$ and that $V_B - V_R - V_C = 0 \implies V_R = V_B - V_C$. Then the voltage across a resistor when charging a capacitor is given by:

$$V_R = V_B - V_C$$

$$V_R = V_B - \left(V_B - V_B \exp\left(-\frac{t}{RC}\right)\right)$$

$$V_t = V_B \exp\left(-\frac{t}{RC}\right)$$

Because current is the same throughout the entire circuit, and because current is the derivative of charge with respect to time, we can differentiate Q_t to obtain I_t . Alternatively, you can use $V = IR$ and divide V_t by R .

$$I_t = \frac{d}{dt}Q_t = \frac{d}{dt}CV_B \left(1 - \exp\left(-\frac{t}{RC}\right)\right)$$

$$I_t = \frac{V_B}{R} \exp\left(-\frac{t}{RC}\right)$$

RC is frequently expressed as τ and is known as a **time constant**. After one time constant, a charging capacitor will have achieved $1 - \frac{1}{e} \times 100\% \approx 63\%$ of its full charge. After one time constant, a discharging capacitor will have only $\frac{1}{e} \times 100\% \approx 37\%$ of its maximum charge left.

	Charging	Discharging
Voltage across Capacitor	$V_C(t) = V_B (1 - e^{-t/\tau})$	$V_C(t) = V_{max} e^{-t/\tau}$
Voltage across Resistor	$V_R(t) = V_B e^{-t/\tau}$	$V_R(t) = V_{max} e^{-t/\tau}$
Current through Resistor	$I(t) = \frac{V_B}{R} e^{-t/\tau}$	$I(t) = \frac{V_{max}}{R} e^{-t/\tau}$
Charge on Capacitor	$Q(t) = CV_B (1 - e^{-t/\tau})$	$Q(t) = Q_{max} e^{-t/\tau}$

For small values of t , the voltage across a charging capacitor is zero, and the current through the circuit is close to the current as if the capacitor were not there $\left(\frac{V_B}{R}\right)$. However, as t increases, the current through the system goes to 0, as if the capacitor were an open switch.

An **empty** capacitor acts as a **closed switch**. A **fully charged** capacitor acts as an **open switch**.