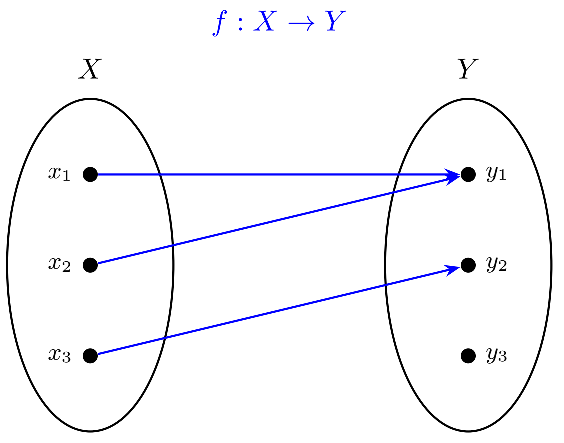
INJECTIVITY, SURJECTIVITY, AND BIJECTIVITY

ALEX GERTNER

A **function** or a **map** is a rule that assigns each element in a set X to some element in a set Y. A function f that maps the elements of a set X to some elements in a set Y is denoted by

$$f : X \to Y$$

If we let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$, we can represent a function f pictorially:

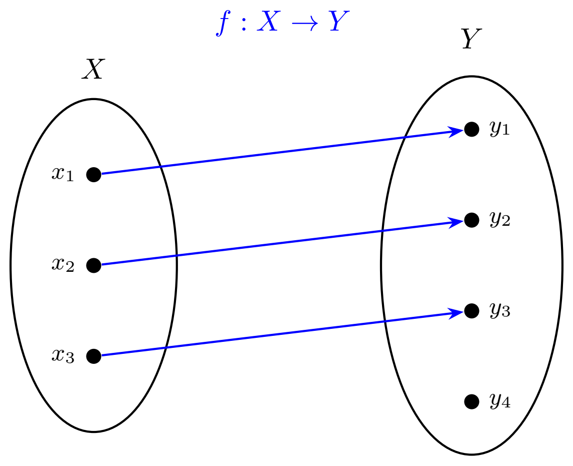


A function is **injective** if no output is hit more than once. That is, no two inputs map to the same output. This can be expressed as:

$$a \neq b \implies f(a) \neq f(b)$$

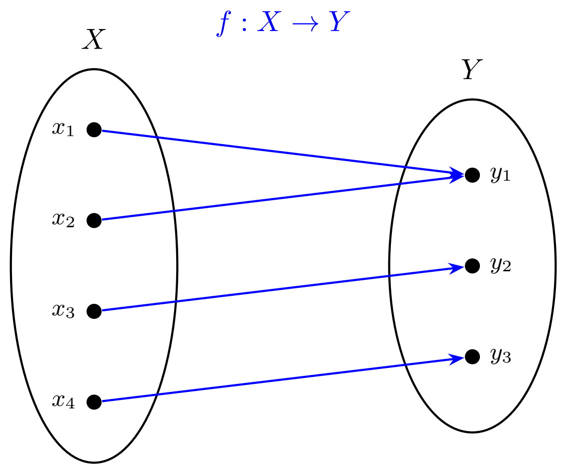
Taking the contrapositive, we see that this is equivalent to

$$f(a) = f(b) \implies a = b$$

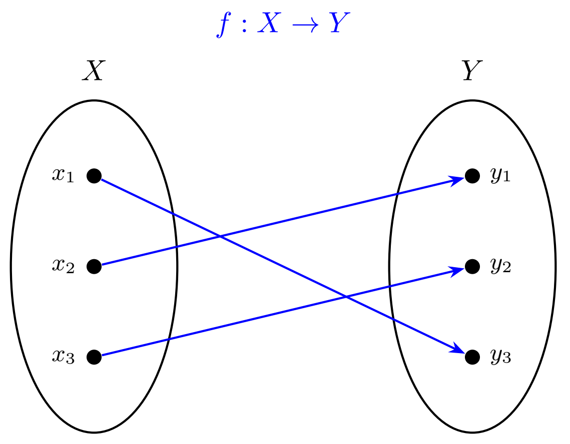


A function is **surjective** if every output is hit at least once. That is, for each element y in the codomain, we can find some x that maps to it.

$$\forall y \in Y \exists x \in X (y = f(x))$$

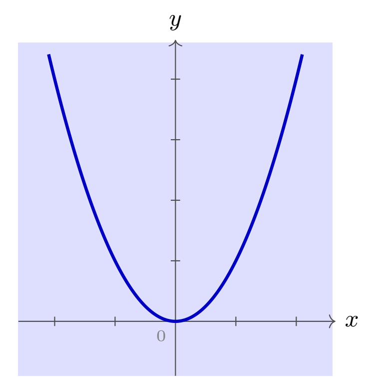


A function is **bijective** if every output is hit at least once and only once. That is, it is both injective and surjective.

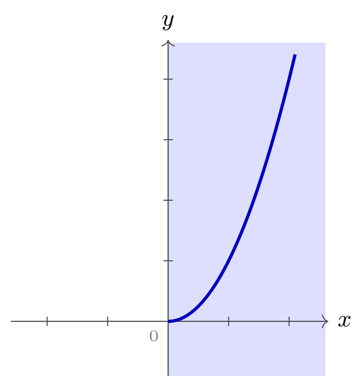


Consider the following four functions.

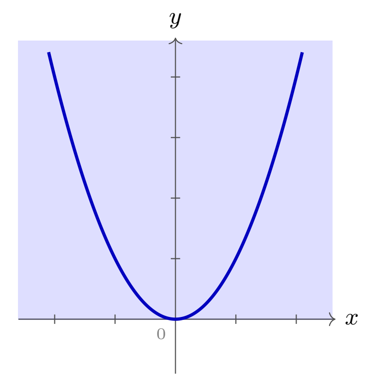
(1) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$
(2) $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = x^2$
(3) $f : \mathbb{R} \to \mathbb{R}^+$ defined by $f(x) = x^2$
(4) $f : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f(x) = x^2$



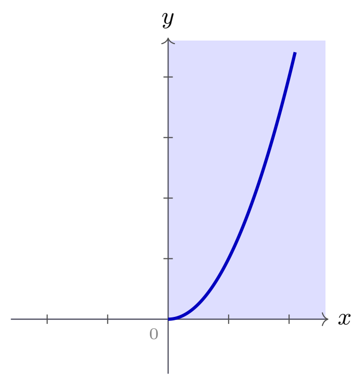
(1) $f : \mathbb{R} \to \mathbb{R}$
neither injective nor surjective



(2) $f : \mathbb{R}^+ \to \mathbb{R}$
injective, not surjective



(3) $f : \mathbb{R} \to \mathbb{R}^+$
surjective, not injective



(4) $f : \mathbb{R}^+ \to \mathbb{R}^+$
bijective

Shading marks the portion of $\mathbb{R} \times \mathbb{R}$ under consideration (domain $\times$ codomain). The curve shows $f(x) = x^2$ restricted to that domain.

Date: December 2024.