ASYMPTOTIC ANALYSIS OF EXPONENTIATION OF LOG OF SINE INTEGRAL

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Abstract. The log of sine integral has been studied since Euler. This paper demonstrates that repeated exponentiation of the integrand behaves like the factorial function.

1. Introduction

This paper will show that the function

$$f(n) = \int_0^{\frac{\pi}{2}} (-\log(\sin(x))^n \, dx$$

is asymptotically equivalent to $\Gamma(n+1)$, where $\Gamma(n)$ is the complete gamma function [2].

$$\Gamma(n) = \int_0^{\infty} e^{-t} t^{n-1} dt$$

2. Notation

We define the binary relation $f(x) \sim g(x)$ to mean that f and g behave the same asymptotically [1].

$$f(x) \sim g(x) \iff \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$$

Equivalently, one could write

$$f(x) \sim g(x) \iff f(x) = g(x)(1 + o(x))$$

3. Lemmas

3.1. Lemma 1.

$$\lim_{x\to\infty} f(x) = 0 \implies \frac{1}{f(x)} \sim \frac{1}{\sin^{-1}(f(x))}$$

Proof

$$\lim_{x\to\infty} f(x) = 0 \implies \lim_{x\to\infty} \frac{\sin^{-1}(f(x))}{f(x)} = \lim_{x\to\infty} \frac{\frac{d}{dx}\sin^{-1}(f(x))}{\frac{d}{dx}f(x)}$$

$$= \lim_{x\to\infty} \frac{\frac{f'(x)}{\sqrt{1-f(x)^2}}}{f'(x)} = \lim_{x\to\infty} \frac{1}{\sqrt{1-f(x)^2}} = 1$$

$$\lim_{x\to\infty} \frac{\sin^{-1}(f(x))}{f(x)} = 1 \implies \frac{1}{f(x)} \sim \frac{1}{\sin^{-1}(f(x))} \qquad \square$$

3.2. **Lemma 2.** *Suppose $f(0)$ and $g(0)$ are positive finite constants, that f and g are positive and monotonic decreasing on $[0, \infty)$, and that the integrals of f and g from 0 to x are convergent for $x \geq 0$. Then*

$$f(x) \sim g(x) \implies \int_0^x f(t)dt \sim \int_0^x g(t)dt$$

Proof

Assume

$$f(x) \sim g(x) \implies f(x) = g(x)(1 + o(1))$$

Consider

$$\lim_{x\to\infty} \frac{\int_0^x f(\tau)d\tau}{\int_0^x g(\tau)d\tau} = \lim_{x\to\infty} \frac{\int_0^x g(\tau)(1+o(1))d\tau}{\int_0^x g(\tau)d\tau} = \lim_{x\to\infty} \frac{\int_0^x g(\tau)d\tau + \int_0^x g(\tau)o(1)d\tau}{\int_0^x g(\tau)d\tau}$$

$$= 1 + \lim_{x\to\infty} \frac{\int_0^x g(\tau)o(1)d\tau}{\int_0^x g(\tau)d\tau} = 1 + \lim_{x\to\infty} o(1) \frac{\int_0^x g(\tau)d\tau}{\int_0^x g(\tau)d\tau} = 1 + o(1) = 0$$

Therefore,

$$\lim_{x\to\infty} \frac{\int_0^x f(\tau)d\tau}{\int_0^x g(\tau)d\tau} = 0 \implies \int_0^x f(t)dt \sim \int_0^x g(t)dt \qquad \square$$

4. Claim

$$\int_0^{\frac{\pi}{2}} (-\log(\sin(x))^n \, dx \sim \Gamma(n+1)$$

5. Proof

By inverting the integrand and integrating horizontally, we obtain

$$\int_0^{\frac{\pi}{2}} (-\log(\sin(x))^n \, dx = \int_0^\infty \sin^{-1}\left(e^{-\sqrt[n]{y}}\right) dy$$

From Lemma 1 and Lemma 2, we obtain

$$\int_0^\infty \sin^{-1}\left(e^{-\sqrt[n]{y}}\right) dy \sim \int_0^\infty e^{-\sqrt[n]{y}} dy$$

Perform the substitution $u = \sqrt[n]{y}$.

$$u = y^{\frac{1}{n}}$$

$$du = \frac{1}{n} y^{\frac{1}{n}-1} dx \implies dx = n y^{\frac{n-1}{n}} du = n(u^n)^{\frac{n-1}{n}} du = n u^{n-1} du$$

$$\int_0^\infty \left(e^{-\sqrt[n]{y}}\right) = n \int_0^\infty e^{-u} u^{n-1} du = n\Gamma(n) = \Gamma(n+1) = n! \qquad \square$$

References

[1] N. G. de Bruijn. *Asymptotic Methods in Analysis*. Dover Publications, 1981.
[2] Eric W. Weisstein. Gamma function. From MathWorld–A Wolfram Web Resource, 2024. Accessed: 2024-12-20.